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Approximability results for stable marriage problems with ties ☆

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Abstract

We consider instances of the classical stable marriage problem in which persons may include ties in their preference lists. We show that, in such a setting, strong lower bounds hold for the approximability of each of the problems of finding an egalitarian, minimum regret and sex-equal stable matching. We also consider stable marriage instances in which persons may express unacceptable partners in addition to ties. In this setting, we prove that there are constants δ, δ' such that each of the problems of approximating a maximum and minimum cardinality stable matching within factors of δ, δ' (respectively) is NP-hard, under strong restrictions. We also give an approximation algorithm for both problems that has a performance guarantee expressible in terms of the number of lists with ties. This significantly improves on the best-known previous performance guarantee, for the case that the ties are sparse. Our results have applications to large-scale centralized matching schemes.

Keywords: Stable marriage problem; Ties; Unacceptable partners; Inapproximability results; Approximation algorithm

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1. Introduction

An instance I of the classical Stable Marriage problem (SM) [7,17,21] involves n men and n women, each of whom ranks all the members of the opposite sex in strict order of preference. A *matching* M in I is a bijection between the men and women. We say that a (man,woman) pair (m,w) *blocks* M, or is a *blocking pair* with respect to M, if each of m and w prefers the other to his/her partner in M. A matching that admits no blocking pair is said to be *stable*. It is known that every instance of SM admits at least one stable matching [3], and that such a matching can be found in $O(n^2)$ time using the Gale/Shapley algorithm [3].

The *man-oriented* version of the Gale/Shapley algorithm [3] yields a stable matching called the *man-optimal stable matching*. This is the unique stable matching in which each man has his best possible partner (and each woman her worst) among all stable matchings. Similarly, the *woman-oriented* version leads to the *woman-optimal stable matching* with analogous optimality conditions for the women (and pessimality conditions for the men).

1.1. "Fair" stable matchings

In view of the fact that man-optimal and woman-optimal stable matchings are woman-pessimal and man-pessimal, respectively, it is of interest to consider stable matchings that are "fair" to both sexes in a precise sense. Given a matching M and a person q in a given SM instance I, define the *cost* of M for q, denoted by $c_M(q)$, to be the ranking of $p_M(q)$ in q's preference list, where $p_M(q)$ denotes q's partner in M. In other words, $c_M(q)$ is one plus the number of persons whom q prefers to $p_M(q)$. Let U and W denote the set of men and women in I, respectively, and let $\mathscr{M}$ denote the set of stable matchings in I. Define an *egalitarian stable matching* to be a stable matching S for which $c(S) = \min_{M\in\mathscr{M}} c(M)$, where $c(M) = \sum_{q\in U\cup W} c_M(q)$ for any $M \in \mathscr{M}$. Similarly, define a *minimum regret stable matching* to be a stable matching S for which $r(S) = \min_{M\in\mathscr{M}} r(M)$, where $r(M) = \max_{q\in U\cup W} c_M(q)$ for any $M \in \mathscr{M}$. Finally, define a *sex-equal stable matching* to be a stable matching S for which $d(S) = \min_{M\in\mathscr{M}} d(M)$, where

$$d(M) = \left| \sum_{m\in U} c_M(m) - \sum_{w\in W} c_M(w) \right|$$

for any $M \in \mathscr{M}$.

Intuitively, an egalitarian stable matching seeks to minimize the total cost of M taken over all persons in I, whilst a minimum regret stable matching aims to minimize the maximum cost of M taken over all persons in I. Finally in a sex-equal stable matching, the total cost of M for the men in I is as close to the total cost of M for the women in I as possible.

Denote the problems of finding an egalitarian, minimum regret and sex-equal stable matching by EGALITARIAN SM, MINIMUM REGRET SM and SEX-EQUAL SM respectively, given an instance of SM. It is known that each of EGALITARIAN SM and MINIMUM REGRET SM is polynomial-time solvable [13,2,6]. However SEX-EQUAL SM has been shown to be NP-hard [16].

1.2. Ties in the preference lists

A natural generalization of SM arises when each person need not rank all members of the opposite sex in *strict* order. Some of those might be indifferent among certain members of the opposite sex, so that preference lists may involve *ties.* [3] We use SMT to stand for the variant of SM in which preference lists may include ties. (Henceforth, we assume that a tie is of length at least two.) In this context, a matching M is stable if there is no (man,woman) pair (m,w), each of whom *strictly* prefers the other to his/her partner in M. [4]

By breaking the ties arbitrarily, an instance I of SMT becomes an instance I' of SM, and clearly a stable matching in I' is also stable in I. Thus a stable matching in I can be found using the Gale/Shapley algorithm. (Conversely, given a stable matching M in I, it is not difficult to see that there is an instance I_M of SM in which M is stable. Hence a matching M is stable in I if and only if M is stable in some instance of SM obtained from I by breaking the ties.)

The stability criterion considered here is referred to as *weak stability* in [11], where two other notions of stability are formulated for SMT, so-called *strong stability* and *super-stability*. However, an instance of SMT need not admit a strongly stable matching or a super-stable matching [11]. By contrast, we have already seen that every instance of SMT admits at least one weakly stable matching. Therefore, perhaps unsurprisingly, of these three definitions, it is weak stability that has received the most attention in the literature [15,18–20]. We are concerned exclusively with weak stability in this paper, and henceforth for brevity, the term *stability* will be used to indicate weak stability when ties are present.

The concept of the cost of a matching for a person may easily be extended to the SMT context. Given a matching M and a person q in an SMT instance I, $c_M(q)$ is the (possibly joint) ranking of $p_M(q)$ in q's preference list. In other words, $c_M(q)$ is one plus the number of persons whom q strictly prefers to $p_M(q)$. Given this extension of the definition of $c_M(q)$, each of the definitions of an egalitarian, minimum regret and sex-equal stable matching in an instance of SMT follows immediately. Define EGALITARIAN SMT, MINIMUM REGRET SMT and SEX-EQUAL SMT to be the analogous problems to EGALITARIAN SM, MINIMUM REGRET SM and SEX-EQUAL SM, respectively, given an instance of SMT.

It is known that each of EGALITARIAN SMT and MINIMUM REGRET SMT is NP-hard, and not approximable within $n^{1-\varepsilon}$, for any $\varepsilon>0$, unless $\mathrm{P}=\mathrm{NP}$, where n is the number of persons in a given SMT instance [18]. In this paper, we improve these results by demonstrating that a worst possible $\Omega(n)$ lower bound on the approximability of each

[3] In this paper, we restrict attention to the case where the indifference takes the form of ties in the preference lists, but the results presented extend to the general case where the preference lists are arbitrary partial orders.

[4] Implicitly here, and henceforth for other stability definitions, such a pair (m,w) is defined to *block* M, or to be a *blocking pair* with respect to M, as for the SM case.

of these problems holds. In addition, we prove that a similar lower bound holds for SEX-EQUAL SMT.

1.3. Unacceptable partners

An alternative natural extension of SM occurs when persons are permitted to express unacceptable partners. We say that person p is *acceptable* to person q if p appears on the preference list of q, and *unacceptable* otherwise. If person q is missing from person p's preference list, p is not prepared to be matched with q, or to form a blocking pair with q. We use SMI to stand for this variant of SM where preference lists may be incomplete.

It follows immediately that a matching M in an instance I of SMI is now a one–one correspondence between a subset of the men and a subset of the women, such that $(m,w)\in M$ implies that each of m,w is acceptable to the other. Also, the revised notion of stability may be defined as follows: M is stable if there is no (man,woman) pair (m,w), each of whom is either unmatched in M and finds the other acceptable, or prefers the other to his/her partner in M. (As a consequence of this definition, it follows that from the point of view of finding stable matchings, we may assume, without loss of generality, that p is acceptable to q if and only if q is acceptable to p.)

A stable matching in I need not be a complete matching. However, all stable matchings in I have the same size, and involve exactly the same men and women [4]. Therefore, each of the definitions of an egalitarian, a minimum regret and a sex-equal stable matching in an instance of SMI follows immediately from its SM definition if we discard the unmatched men and women from consideration. In addition, it is a simple matter to extend the Gale/Shapley algorithm to the SMI setting (see [7, Section 1.4.2]).

1.4. Ties and unacceptable partners

The variant of the stable marriage problem which incorporates *both* extensions described above is denoted SMTI. Thus, an instance I of SMTI comprises preference lists, each of which may involve ties and/or unacceptable partners. A combination of the earlier definitions indicates that a matching M in I is stable if there is no (man,woman) pair (m,w), each of whom is either unmatched in M and finds the other acceptable, or strictly prefers the other to his/her partner in M.

As observed above, all stable matchings for a given instance of SMI are of the same size, and all stable matchings for a given instance of SMT are complete (and therefore of the same size). However, for a given instance of SMTI, it is no longer the case that all stable matchings need be of the same size [18]. Furthermore, each of the problems of finding a stable matching of maximum or minimum size, given an SMTI instance, is NP-hard [15,18]. Therefore, one is naturally led to consider the approximability properties of each of these problems. It turns out that each problem admits an approximation algorithm with a performance ratio of 2, since the size of any stable matching is at least half the size of a maximum cardinality stable matching and is at most twice the size of a minimum cardinality stable matching [18]. This has left open the question of whether better approximation algorithms for these problems exist.

In this paper, we present both positive and negative results regarding the approximability of each of these problems: we show that the existence of a polynomial-time approximation scheme (PTAS) for either of these problems is unlikely, since there exist constants δ, δ' such that approximating each problem within a factor of δ, δ' (respectively) is NP-hard, under strong restrictions on the instance. However, we also show that, for a given SMTI instance I, the difference in size between a maximum and a minimum cardinality stable matching is bounded by $t(I)$, the number of preference lists that contain ties, and this leads to an approximation algorithm for both problems with a performance guarantee dependent on $t(I)$. When $t(I)$ is relatively small compared to the size of the instance, our result significantly improves on the best-known previous result regarding the approximability of both problems, namely the performance ratio of 2.

1.5. Practical applications

The problems of finding "fair" stable matchings and maximum cardinality stable matchings in a given instance of SMTI have particular significance in practical applications. In a number of countries, large-scale automated matching schemes produce stable matchings of graduating medical students to hospital posts based on the preferences of students over hospitals and vice versa. Examples of such schemes are the National Resident Matching Program (NRMP) [20] in the U.S., the Canadian Resident Matching Service (CaRMS) [1] and the Scottish Pre-registration house officer Allocation scheme (SPA) [12].

The algorithms employed by the NRMP and CaRMS essentially solve a many-one generalization of SMI called the Hospitals/Residents problem (HR) [7, Section 1.6]. In the context of these two matching schemes, hospitals must rank a possibly large number of applicants in strict order of preference. However, it is unrealistic to expect large and popular hospitals to provide a strict ranking of all of their applicants. The SPA scheme permits hospitals to include ties, a situation which may be modelled by a many-one matching problem known as the Hospitals/Residents problem with Ties (HRT) [14], a generalization of each of HR and SMTI.

Thus, since the stable matchings in an instance of SMTI may be of different sizes, the same is true for HRT. Yet a prime objective of any matching scheme must be to match as many applicants as possible, and hence this motivates the search for large stable matchings. In addition, administrators of matching schemes may be interested to find stable matchings that are as fair as possible for both applicants and hospitals alike, and hence this motivates the search for egalitarian, minimum regret and sex-equal stable matchings. Thus our approximability results have implications for matching schemes such as SPA.

1.6. Organization of the paper

The remainder of this paper is organized as follows. In Section 2 we prove that it is hard to approximate the MIN MAXIMAL MATCHING optimization problem (defined in that section) in a certain class of graphs. This result is required in order to establish, in Section 3, the hardness results for the problems of approximating a maximum or minimum cardinality stable matching in a given instance of SMTI. Then, in Section 4 we present the approximation algorithm for the variants of these problems where, in a given SMTI instance, the number of lists containing ties is bounded. The $\Omega(n)$ lower bounds for each of the problems of approximating EGALITARIAN SMT, MINIMUM REGRET SMT and SEX-EQUAL SMT are presented in Section 5. Finally, in Section 6 we present some concluding remarks.

2. Hardness of approximating MIN MAXIMAL MATCHING

We begin this section with some graph-related definitions. Given a graph $G=(V,E)$, a *strongly stable set* S is a subset of V such that the distance between every pair of vertices in S is at least 3. A matching M in G is *maximal* if no proper superset of M is a matching in G. Let $\beta_0(G)$, $\beta_{SS}(G)$ and $\beta_1^-(G)$ denote, respectively, the sizes of a maximum independent set, a maximum strongly stable set and a minimum maximal matching in G. Define MIN MAXIMAL MATCHING to be the problem of computing $\beta_1^-(G)$, given a graph G.

MIN MAXIMAL MATCHING is NP-hard, even for subdivision graphs of graphs of maximum degree 3 [10] (given a graph G, the *subdivision graph* of G, denoted by $S(G)$, is obtained by subdividing each edge $\{u,w\}$ of G in order to obtain two edges $\{u,v\}$ and $\{v,w\}$ of $S(G)$, where v is a new vertex). In this section, we will establish that MIN MAXIMAL MATCHING is hard to approximate in a certain graph class; this result will be required in the next section. In particular, we will prove the following:

Theorem 1. *It is NP-hard to approximate* MIN MAXIMAL MATCHING *within* δ_0, *for some* $\delta_0>1$. *The result holds even if the instance is restricted to be the subdivision graph of some cubic graph.*

Our proof of Theorem 1 involves a chain of reductions starting from MAX-IS. This is the problem of computing $\beta_0(G)$, given a graph G. We denote by MAX-IS(k) the restriction of MAX-IS in which G is regular of degree k.

Theorem 2 (Halldórsson and Yoshihara [9]). *It is NP-hard to approximate* MAX-IS(3) *within* δ_1, *for some* $\delta_1<1$.

In fact, there exists a constant $c_1>0$ such that it is NP-hard to distinguish between instances $G=(V,E)$ of MAX-IS(3) such that $\beta_0(G)\geqslant c_1 n$ and $\beta_0(G)<\delta_1 c_1 n$, where $n=|V|$.

We will use Theorem 2 together with the notion of a *gap-preserving reduction* [22, p. 308], which may be defined as follows:

Definition 3. Let Π_1 and Π_2 be two optimization problems. Denote by $OPT_i(x)$ the optimal measure over all feasible solutions for a given instance x of Π_i ($i\in\{1,2\}$). Let α be some constant ($\alpha\leqslant 1$ if Π_1 is a maximization problem; $\alpha\geqslant 1$ otherwise), and let g_1 be a function that maps an instance x of Π_1 to a positive rational number. Then a *gap-preserving reduction* from Π_1 to Π_2 is a tuple $\langle f, \beta, g_2 \rangle$ such that:

- f maps an instance x of Π_1 to an instance $f(x)$ of Π_2 in polynomial time;
- β is a constant ($\beta \leqslant 1$ if Π_2 is a maximization problem; $\beta \geqslant 1$ otherwise);
- g_2 maps an instance $f(x)$ of Π_2 to a positive rational number;
- if Π_1 and Π_2 are maximization problems, then for any instance x of Π_1:
  - if $OPT_1(x) \geqslant g_1(x)$, then $OPT_2(f(x)) \geqslant g_2(f(x))$;
  - if $OPT_1(x) < \alpha g_1(x)$, then $OPT_2(f(x)) < \beta g_2(f(x))$;

  (if Π_i is a minimization problem, for $i \in \{1,2\}$, then the two inequalities involving OPT_i in the above conditions should be reversed).

The following proposition is an immediate consequence of Definition 3.

Proposition 4. *Let Π_1 and Π_2 be two maximization problems, and suppose that there is a gap-preserving reduction from Π_1 to Π_2. Assuming the notation of Definition* 3, *suppose further that it is NP-hard to distinguish between instances x of Π_1 such that $OPT_1(x) \geqslant g_1(x)$ and $OPT_1(x) < \alpha g_1(x)$. Then it is NP-hard to distinguish between instances $f(x)$ of Π_2 such that $OPT_2(f(x)) \geqslant g_2(f(x))$ and $OPT_2(f(x)) < \beta g_2(f(x))$.* (*If Π_i is a minimization problem, for $i \in \{1,2\}$, then the two inequalities involving OPT_i in the above conditions should be reversed*). *Hence it is NP-hard to approximate Π_2 within β.*

Our first gap-preserving reduction involves MAX-SSS. This is the problem of computing $\beta_{SS}(G)$ for a given graph G. We denote by MAX-SSS(k) the restriction of MAX-SSS in which G is regular of degree k.

Theorem 5. *It is NP-hard to approximate* MAX-SSS(3) *within δ_2, for some $\delta_2 < 1$.*

Proof. Let $G = (V, E)$ be a cubic graph, given as an instance of MAX-IS(3), where $n = |V|$ and $m = |E|$. We construct a cubic graph $G' = (V', E')$ as an instance of MAX-SSS(3) as follows. As in the proof of Corollary 3.4 of [10], we initially replace every edge $\{v, w\}$ of G by a component comprising the edges $\{v, u\}, \{u, w\}, \{u, u'\}, \{u', u''\}$. This leaves m vertices of degree 1 in G' and m vertices of degree 2 in G'.

We may eliminate such vertices as follows. To every vertex v of degree 1 in G', connect the component shown in Fig. 1(a). Similarly, for every vertex v of degree 2 in G', connect the component shown in Fig. 1(b). It is then clear that the modified graph G' is cubic.

It is straightforward to verify that G has an independent set of size k if and only if G' has a strongly stable set of size $3m + k$, and hence $\beta_{SS}(G') = \beta_0(G) + 3m$. Now $2m = 3n$ as G is cubic, and it may be verified that $n' = 22n$, where $n' = |V'|$.

Now let c_1 and δ_1 be the constants given by Theorem 2, such that it is NP-hard to distinguish between the cases $\beta_0(G) \geqslant c_1 n$ and $\beta_0(G) < \delta_1 c_1 n$. Hence if $\beta_0(G) \geqslant c_1 n$, then $\beta_{SS}(G') \geqslant c_2 n'$, whilst if $\beta_0(G) < \delta_1 c_1 n$, then $\beta_{SS}(G') < \delta_2 c_2 n'$, where $c_2 = (2c_1 + 9)/44$ and $\delta_2 = (2\delta_1 c_1 + 9)/(2c_1 + 9)$. The result then follows by Theorem 2 and Proposition 4. $\square$

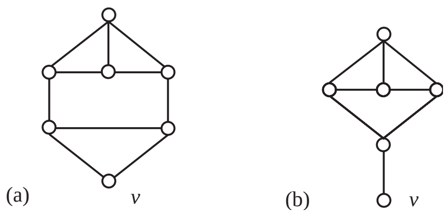

Fig. 1. Components attached to vertices of degree 1 or 2 in G'.

Our second gap-preserving reduction is sufficient to prove Theorem 1.

Proof of Theorem 1. Let $G=(V,E)$ be a cubic graph, given as an instance of MAX-SSS(3), where $n=|V|$ and $m=|E|$. The constructed instance of MIN MAXIMAL MATCHING is $S(G)$ (recall that $S(G)$ is the subdivision graph of G). Now by Lemmas 3.1 and 3.2 of [10], G has a strongly stable set of size k if and only if $S(G)$ has a maximal matching of size $n-k$. Thus it follows that $\beta_1^-(S(G))+\beta_{SS}(G)=n$. Now $2m=3n$ as G is cubic, and $m'=2m$, where m' is the number of edges of $S(G)$.

Now let c_2 and δ_2 be the constants given by Theorem 5, such that it is NP-hard to distinguish between the cases $\beta_{SS}(G)\geqslant c_2 n$ and $\beta_{SS}(G)<\delta_2 c_2 n$. Hence if $\beta_{SS}(G)\geqslant c_2 n$, then $\beta_1^-(S(G))\leqslant c_0 m'$, whilst if $\beta_{SS}(G)<\delta_2 c_2 n$, then $\beta_1^-(S(G))>\delta_0 c_0 m'$, where $c_0=(1-c_2)/3$ and $\delta_0=(1-\delta_2 c_2)/(1-c_2)$. The result then follows by Theorem 5 and Proposition 4. □

3. Hardness of approximating MAX SMTI and MIN SMTI

Given an instance I of SMTI, let $s^+(I)$ (respectively, $s^-(I)$) denote the size of a maximum (respectively, minimum) cardinality stable matching in I. Define MAX (respectively, MIN) SMTI to be the problem of computing $s^+(I)$ (respectively, $s^-(I)$), given an SMTI instance I.

Each of MAX SMTI and MIN SMTI is NP-hard [15,18]. In this section, we prove that there exist constants δ, δ' such that each of the problems of approximating MAX SMTI and MIN SMTI within a factor of δ, δ' (respectively) is NP-hard. In each case, the result holds under the restriction that the ties belong to the preference lists of one sex only, and preference lists have constant length. We begin by considering MAX SMTI.

Theorem 6. *It is NP-hard to approximate* MAX SMTI *within* δ_3, *for some* $\delta_3<1$. *The result holds even if the preference lists in the given instance are of constant length, there is at most one tie per list, and the ties occur on one side only.*

Proof. Let $G=(V,E)$ be the subdivision graph of some cubic graph, given as an instance of MIN MAXIMAL MATCHING. Then G has a bipartition of V into the left-hand vertex set U and the right-hand vertex set W, where every vertex in U has degree 3 and every vertex in W has degree 2.

Let $U=\{m_1,m_2,\ldots,m_s\}$ and $W=\{w_1,w_2,\ldots,w_t\}$. For each i ($1\leqslant i\leqslant s$), assume that m_i is adjacent in G to the vertices in W_i, where $W_i=\{w_{k_{3i-2}},w_{k_{3i-1}},w_{k_{3i}}\}$. Also, assume that p_j and q_j are two sequences such that $p_j<q_j$, $\{m_{p_j},w_j\}\in E$ and $\{m_{q_j},w_j\}\in E$ ($1\leqslant j\leqslant t$).

We form an instance I of MAX SMTI as follows. Let $\mathscr{U}$ be the set of men in I, where $\mathscr{U}=U\cup X\cup Z$, $X=\{x_1,x_2,\ldots,x_t\}$, and $Z=\{z_1,z_2,\ldots,z_t\}$. Also, let $\mathscr{W}$ be the set of women in I, where $\mathscr{W}=W\cup W'\cup Y$, $W'=\{w'_1,w'_2,\ldots,w'_t\}$, and $Y=\{y_1,y_2,\ldots,y_s\}$. For each i ($1\leqslant i\leqslant s$), let $W'_i=\{w'_{k_{3i-2}},w'_{k_{3i-1}},w'_{k_{3i}}\}$. Clearly $|\mathscr{U}|=|\mathscr{W}|=s+2t$. Create a preference list for each person in I as follows:

$$\begin{array}{lll@{\qquad}lll}
m_i: & (W_i\cup W'_i)\ y_i & (1\leqslant i\leqslant s), & w_j: & z_j\ m_{p_j}\ m_{q_j}\ x_j & (1\leqslant j\leqslant t),\\
x_i: & w_i & (1\leqslant i\leqslant t), & w'_j: & z_j\ m_{q_j}\ m_{p_j} & (1\leqslant j\leqslant t),\\
z_i: & (w_i\ w'_i) & (1\leqslant i\leqslant t), & y_j: & m_j & (1\leqslant j\leqslant s).
\end{array}$$

Note that, in a given preference list throughout this paper, persons listed within round brackets are tied. Clearly the ties in I occur in the men's preference lists only and there is at most one tie per list. Also each man's list has length at most 7, whilst each woman's list has length at most 4.

Suppose that M is a maximal matching in G, where $|M|=\beta_1^-(G)$. We construct a matching M' in I as follows. For each i ($1\leqslant i\leqslant s$), suppose firstly that m_i is matched in M, to w_j say ($1\leqslant j\leqslant t$). If $i=p_j$, add the pairs (m_i,w_j) and (z_j,w'_j) to M'. If $i=q_j$, add the pairs (m_i,w'_j) and (z_j,w_j) to M'.

On the other hand, if m_i is unmatched, add the pair (m_i,y_i) to M'.

Finally, for any j ($1\leqslant j\leqslant t$), if w_j is unmatched, add the pairs (x_j,w_j) and (z_j,w'_j) to M'.

Clearly M' is a matching in I, and $|M'|=2|M|+(s-|M|)+2(t-|M|)=s+2t-|M|$. It is straightforward to verify that no man in $X\cup Z$ can belong to a blocking pair of M'. Now suppose that (m_i,w) blocks M' for some i ($1\leqslant i\leqslant s$) and $w\in\mathscr{W}$. Then $(m_i,y_i)\in M'$, so that $w=w_j$ for some j ($1\leqslant j\leqslant t$) and $(x_j,w_j)\in M'$. Thus each of m_i and w_j is unmatched in M, and $\{m_i,w_j\}\in E$. Thus $M\cup\{\{m_i,w_j\}\}$ is a matching in G, contradicting the maximality of M. Hence M' is stable in I. Also $s^+(I)\geqslant s+2t-|M|=s+2t-\beta_1^-(G)$.

Conversely, suppose that M' is a stable matching in I, where $|M'|=s^+(I)$. For each j ($1\leqslant j\leqslant t$), either $(z_j,w_j)\in M'$ or $(z_j,w'_j)\in M'$, for otherwise (z_j,w_j) blocks M'. Hence

$$M=\left\{\{m_i,w_j\}:\begin{array}{l}(1\leqslant i\leqslant s)\wedge(1\leqslant j\leqslant t)\wedge\\ ((m_i,w_j)\in M'\vee(m_i,w'_j)\in M')\end{array}\right\}$$

is a matching in G. Also $|M'|\leqslant|M|+(t-|M|)+t+(s-|M|)=s+2t-|M|$, for every edge in M contributes one (man,woman) pair to M', and in addition, at most $(t-|M|)$ men in X can be matched in M', exactly t men in Z are matched in M', and at most $(s-|M|)$ women in Y can be matched in M' (and everybody who could be matched in M' has now been counted).

Suppose that M is not maximal. Then there is some edge $\{m_i, w_j\}$ in G such that no edge of M is incident to either m_i or w_j. Thus by definition of M, either m_i is unmatched in M' or $(m_i, y_i) \in M'$. Similarly, either (i) $(x_j, w_j) \in M'$ or w_j is unmatched, or (ii) w'_j is unmatched in M'. In case (i) (m_i, w_j) blocks M', whilst in case (ii) (m_i, w'_j) blocks M', a contradiction. Hence M is a maximal matching in G, and $s^+(I) = |M'| \leqslant s + 2t - |M| \leqslant s + 2t - \beta_1^-(G)$.

Hence $s^+(I) + \beta_1^-(G) = s + 2t$. Now $2t = 3s$, as G is the subdivision graph of some cubic graph. Also $n = s + 2t$ and $m = 2t$, where n is the number of men in I and m is the number of edges of G.

Let c_0 and δ_0 be the constants given by Theorem 1, such that it is NP-hard to distinguish between the cases $\beta_1^-(G) \leqslant c_0 m$ and $\beta_1^-(G) > \delta_0 c_0 m$. Hence if $\beta_1^-(G) \leqslant c_0 m$, then $s^+(I) \geqslant c_3 n$, whilst if $\beta_1^-(G) > \delta_0 c_0 m$, then $s^+(I) < \delta_3 c_3 n$, where $c_3 = (4 - 3c_0)/4$ and $\delta_3 = (4 - 3\delta_0 c_0)/(4 - 3c_0)$. The result then follows by Theorem 1 and Proposition 4. □

We now demonstrate how to modify the proof of Theorem 6 in order to establish the hardness of approximating MIN SMTI under the same restrictions.

Theorem 7. *It is NP-hard to approximate* MIN SMTI *within δ_4, for some $\delta_4 > 1$. The result holds even if the preference lists in I are of constant length, there is at most one tie per list, and the ties occur on one side only.*

Proof. The gap-preserving reduction is similar to the one given by the proof of Theorem 6, with some small modifications. In the constructed instance I, the set of men and women no longer includes the persons in $X \cup Y$. Any such person is now removed from the preference list of any remaining person in I. Now each man's preference list is of length at most 6 and each woman's preference list is of length at most 3.

Suppose firstly that M is a maximal matching in G, where $|M| = \beta_1^-(G)$. The construction of the matching M' in I is similar to the previous one; the only difference is as follows. If m_i is unmatched in M, no pair is added to M', whilst if w_j is unmatched in M, the pair (z_j, w_j) is added to M'. It is straightforward to verify that M' is a stable matching in I and $s^-(I) \leqslant |M'| = t + |M| = t + \beta_1^-(G)$.

Conversely, suppose that M' is a stable matching in I, where $|M'| = s^-(I)$. Then using a similar argument to before we may construct a maximal matching M in G, where $s^-(I) = |M'| = t + |M| \geqslant t + \beta_1^-(G)$.

Hence $s^-(I) = t + \beta_1^-(G)$. Now $2t = 3s$, as G is the subdivision graph of some cubic graph. Also $n = s + t$ and $m = 2t$, where n is the number of men in I and m is the number of edges of G.

Let c_0 and δ_0 be the constants given by Theorem 1, such that it is NP-hard to distinguish between the cases $\beta_1^-(G) \leqslant c_0 m$ and $\beta_1^-(G) > \delta_0 c_0 m$. Hence if $\beta_1^-(G) \leqslant c_0 m$, then $s^-(I) \leqslant c_4 n$, whilst if $\beta_1^-(G) > \delta_0 c_0 m$, then $s^-(I) < \delta_4 c_4 n$, where $c_4 = 3(1 + 2c_0)/5$ and $\delta_4 = (1 + 2\delta_0 c_0)/(1 + 2c_0)$. The result then follows by Theorem 1 and Proposition 4. □

It follows immediately from Theorems 6 and 7 that neither MAX SMTI nor MIN SMTI admits a polynomial-time approximation scheme unless P = NP.

4. Approximation algorithm for MAX SMTI and MIN SMTI

As observed earlier, it is shown in [18] that a maximum cardinality stable matching can have size at most twice that of a minimum cardinality stable matching. Hence, the obvious polynomial-time algorithm for SMTI—break all ties in an arbitrary way and apply the classical Gale/Shapley algorithm to the resulting instance of SMI—is simultaneously an approximation algorithm for both MAX and MIN SMTI with a performance ratio of 2.

There is no known approximation algorithm for either problem with a stronger performance ratio, even for special cases of the problems in which the ties are restricted to one side, or to the tails of the preference lists. A case of particular interest arises when there is a limit on the number of preference lists that contain ties, and in this section we show that some progress can be made in establishing additional approximation bounds in this setting.

Ideally, in the case of MAX SMTI, one might hope for a bound of the form $s^+(I)/|M| \leqslant f(p)$ given an instance I of SMTI, where M is a stable matching found by some approximation algorithm (or just any stable matching, found by breaking ties arbitrarily), p is the proportion of preference lists that contain ties, and $f(p)$ is a function that decreases to 1 as p decreases to 0.

However, it is not hard to see that a bound of this form is infeasible. Were such an algorithm to exist, a 'gap' argument could be used to show that it could solve instances of MAX SMTI exactly. Given an arbitrary such instance, it could be 'expanded' by the addition of new persons, none of whom has a tie in his or her list, and none of whom can be matched in any stable matching. With an appropriate expansion factor, application of the supposed approximation algorithm to this derived instance would solve the original instance exactly.

Instead we derive a bound on the *difference* in size between a maximum (or minimum) cardinality stable matching and an arbitrary stable matching, expressed in terms of the number of preference lists that contain ties. So the usual approximation algorithm —break all ties arbitrarily and apply the Gale/Shapley algorithm—has a performance guarantee, for both MAX SMTI and MIN SMTI, expressible as a difference rather than a ratio. As observed by Garey and Johnson [5, pp. 137–138], this form of performance guarantee can reasonably be viewed as being stronger than the more familiar performance ratio form, and there are relatively few NP-hard problems for which approximation algorithms with performance guarantees of this kind are known.

Some additional definitions are necessary before presenting the main results of this section. Let M and M' be stable matchings for an instance I of SMTI. If a person p strictly prefers his partner in M to his partner in M', or is matched in M but not in M', then we say that *p strictly prefers M to M'*. If p is indifferent between his partners in M and M', or has the same partner in M as in M', or is matched in neither M nor M', then we say that *p is indifferent between M and M'*. Define a *tied pair* to be a pair (m,w) such that m is in a tie in w's list, or w is in a tie in m's list (or both). In what follows, $tp(M)$ denotes the number of tied pairs in M, and $t(I)$ denotes the number of preference lists in I that contain ties. In general $tp(M)$ depends on the matching M, whilst $t(I)$ is invariant for the given instance I; clearly $tp(M) \leqslant t(I)$. [5]

Lemma 8. *Let T be a maximum cardinality stable matching for a given instance I of* SMTI. *Then if M is an arbitrary stable matching in I, $|T| \leqslant |M| + tp(M)$.*

Proof. We construct an undirected graph $G = G(M,T)$ as follows: G has a vertex for each person in I, and two vertices are joined by a blue (respectively, red) edge if the corresponding persons are matched in T but not in M (respectively in M but not in T). It is clear that the connected components of G are paths and cycles with edges of alternating colour. Furthermore, $|T| - |M|$ is at most equal to the number of *blue augmenting paths* in G, i.e., the number of paths of odd length in which the first and last edges are blue. Further, every such path has at least three edges, since a component that is a path of length one would provide a blocking pair for one of the supposed stable matchings.

We claim that, in every blue augmenting path, at least one of the intermediate vertices represents a person who is indifferent between T and M, and is therefore in a tied pair in both T and M. This claim, together with the preceding observation, suffices to establish the lemma.

To establish the claim, let $p_1, q_1, \ldots, p_r, q_r$ form a blue augmenting path in G, for some $r \geqslant 2$. Since p_1 and q_r are both matched in T but not in M, they both strictly prefer T to M. Suppose that no person in the path is indifferent between T and M. A simple inductive proof starting from p_1 then reveals that q_i $(i = 1, 2, \ldots, r-1)$ strictly prefers M to T, otherwise (p_i, q_i) would block M, and p_i $(i = 2, 3, \ldots, r)$ strictly prefers T to M, otherwise (p_i, q_{i-1}) would block T. Thus (p_r, q_r) blocks M, a contradiction. Hence at least one of the p_i $(2 \leqslant i \leqslant r)$ or q_i $(1 \leqslant i \leqslant r-1)$ must be indifferent between T and M, as claimed. □

Since $tp(M) \leqslant |M|$, it follows immediately by Lemma 8 that there exists an approximation algorithm for MAX SMTI with performance ratio 2. Using a similar argument to the one employed in the proof of Lemma 8, we may deduce that $|M| \leqslant |S| + tp(S)$, where S is a stable matching of minimum cardinality. Since $tp(S) \leqslant |S|$, it follows immediately that there exists an approximation algorithm for MIN SMTI, also with performance ratio 2. The inequality established by Lemma 8 also leads to the following result:

Theorem 9. *There is an approximation algorithm A such that, given any instance I of either* MAX SMTI *or* MIN SMTI, *A finds a stable matching M in I satisfying the following*

[5] The results of this section may be extended to the case that preference lists are partially ordered by making the following amendments to two key definitions. In this setting, define a *tied pair* to be a pair (m,w) such that w is indifferent between m and some other man, or m is indifferent between w and some other woman (or both). Define $t(I)$ to be the number of preference lists that are not linearly ordered.

inequality:

$$s^+(I) - t(I) \leqslant |M| \leqslant s^-(I) + t(I).$$

Additionally, we have that $s^+(I) \leqslant s^-(I) + t(I)$.

Proof. Let M be defined as in Lemma 8. Since $tp(M) \leqslant t(I)$, Lemma 8 implies that $s^+(I) - t(I) \leqslant |M| \leqslant s^+(I)$. Also by Lemma 8, $s^+(I) \leqslant s^-(I) + t(I)$, and hence the result follows. □

We remark that, when the ties in a given instance I of SMTI are sparse, i.e. $t(I)$ is small compared to the numbers of men and women in I, the performance guarantee indicated by Theorem 9 is a significant improvement on the best-known previous result, namely the 2-approximation algorithm for each of MAX SMTI and MIN SMTI.

The following instance is an illustration of the worst case for the above theorem. For each $n \geqslant 1$, we define an SMTI instance I with $2n$ men, namely $\{p_1, \ldots, p_n, q_1, \ldots, q_n\}$, and $2n$ women, namely $\{r_1, \ldots, r_n, s_1, \ldots, s_n\}$. For each i $(1 \leqslant i \leqslant n)$, define preference lists for p_i, q_i, r_i, s_i as follows:

$$\begin{array}{llll} p_i : & s_i \; r_i & \qquad r_i : & p_i \\ q_i : & s_i & \qquad s_i : & (p_i \; q_i) \end{array}$$

There is a stable matching of size n (namely $M_1 = \{(p_i, s_i):\ 1 \leqslant i \leqslant n\}$) and one of size $2n$ (namely $M_2 = \{(p_i, r_i), (q_i, s_i):\ 1 \leqslant i \leqslant n\}$). Clearly $s^+(I) = 2n$, and also $s^-(I) = n$ since $|M_2| = 2|M_1|$. Since the difference between $s^+(I)$ and $s^-(I)$ is the number of lists with ties, the bounds given by Theorem 9 are tight.

5. "Fair" stable matchings in SMT

In this section, we give $\Omega(n)$ lower bounds for the approximability of EGALITARIAN SMT, MINIMUM REGRET SMT and SEX-EQUAL SMT in an instance of SMT with n men and n women. We begin by considering EGALITARIAN SMT. Note that, for any matching M in such an instance of SMT, it follows that $2n \leqslant c(M) \leqslant 2n^2$. Hence an approximation algorithm with performance guarantee n is trivial. Our inapproximability result is therefore optimal within a constant factor.

Theorem 10. *It is NP-hard to approximate* EGALITARIAN SMT *within* δn, *for some* $\delta > 0$, *where* n *is the number of men in a given* SMT *instance.*

Proof. We give a reduction from an instance I of MAX SMTI as constructed by the proof of Theorem 6. One property of I is that there exists a constant d such that the length of each preference list in I is at most d. Let c_3 and δ_3 be the constants given by Theorem 6, such that it is NP-hard to distinguish the cases $s^+(I) \geqslant c_3 n$ and $s^+(I) < \delta_3 c_3 n$, where n is the number of men in I.

Let $X = \{m_1, m_2, \ldots, m_n\}$ be the set of men in I and let $Y = \{w_1, w_2, \ldots, w_n\}$ be the set of women of I. For each i $(1 \leqslant i \leqslant n)$, let P_i and Q_i denote the preference lists of

m_i and w_i in I, respectively. We call the women in P_i *proper women* for m_i, and we call the men in Q_i *proper men* for w_i.

We transform I into an instance I' of EGALITARIAN SMT as follows. Let $U = X \cup X'$ and $W = Y \cup Y'$ be the sets of men and women in I', respectively, where $X' = \{m'_1, m'_2, \ldots, m'_{(1-c_3)n}\}$ and $Y' = \{w'_1, w'_2, \ldots, w'_{(1-c_3)n}\}$. The preference lists in I' are constructed as follows:

$$\begin{array}{ll} m_i : P_i\ (Y')\ [Y \backslash P_i] & (1 \leqslant i \leqslant n), \\ m'_i : (W) & (1 \leqslant i \leqslant (1-c_3)n), \\ w_i : Q_i\ (X')\ [X \backslash Q_i] & (1 \leqslant i \leqslant n), \\ w'_i : (U) & (1 \leqslant i \leqslant (1-c_3)n). \end{array}$$

Note that, in a given person's preference list, persons within square brackets are listed in arbitrary strict order where the symbol appears.

Suppose firstly that I has a stable matching M such that $|M| \geqslant c_3 n$. Then there is a set $X_u \subseteq X$ of men who are unmatched in M, where $|X_u| \leqslant (1-c_3)n$. Similarly there is a set $Y_u \subseteq Y$ of women who are unmatched in M, where $|Y_u| \leqslant (1-c_3)n$. Let M_1 be a matching that assigns each man in X_u to a woman in Y', and let M_2 be a matching that assigns each woman in Y_u to a man in X'. Now let M_3 be a perfect matching of the remaining unmatched members of X' and Y'. Finally, let $M' = M \cup M_1 \cup M_2 \cup M_3$. It may be verified that M' is a stable matching in I', and

$$\begin{aligned} c(M') &\leqslant 2n(d+1) + 2(1-c_3)n \\ &\leqslant 2n(d+2). \end{aligned}$$

On the other hand, suppose $s^+(I) < \delta_3 c_3 n$. Now let M' be any stable matching in I'. Then $< \delta_3 c_3 n$ men in X are matched in M' to one of their proper women. Now at most $(1-c_3)n$ of the remaining men in X can be matched to a woman in Y'. Hence there are $> c_3 n(1-\delta_3)$ men u in X such that $c_{M'}(u) > (1-c_3)n$. Similarly there are $> c_3 n(1-\delta_3)$ women w in Y such that $c_{M'}(w) > (1-c_3)n$. Hence $c(M') > 2\varepsilon n^2$, where $\varepsilon = c_3(1-c_3)(1-\delta_3)$.

Therefore by Theorem 6, it is NP-hard to approximate EGALITARIAN SMT within $\varepsilon n/(d+2)$. □

We now consider MINIMUM REGRET SMT. Note that, for any matching M in an instance of SMT with n men and n women, it follows that $1 \leqslant r(M) \leqslant n$. Hence, an approximation algorithm with performance guarantee n is trivial. Therefore again, the $\Omega(n)$ lower bound that we establish is optimal within a constant factor.

Theorem 11. *It is NP-hard to approximate* MINIMUM REGRET SMT *within* δn, *for some* $\delta > 0$, *where n is the number of men in a given* SMT *instance.*

Proof. We use the same reduction as described in the proof of Theorem 10. Let I, I', n, c_3, δ_3 and d be as above. If $s^+(I) \geqslant c_3 n$, then I' has a stable matching M' such that $r(M') \leqslant d+1$. On the other hand, if $s^+(I) < \delta_3 c_3 n$ then in any stable matching M' in I', at least one man $u \in X$ satisfies $c_{M'}(u) > (1-c_3)n$. Hence $r(M') > (1-c_3)n$.

Therefore by Theorem 6, it is NP-hard to approximate MINIMUM REGRET SMT within $(1-c_3)n/(d+1)$. □

The final problem that we consider in this section is SEX-EQUAL SMT. We establish an inapproximability result for this problem similar to those of Theorems 10 and 11.

Theorem 12. *It is NP-hard to approximate* SEX-EQUAL SMT *within* δn, *for some* $\delta > 0$, *where* n *is the number of men in a given* SMT *instance.*

Proof. We formulate a reduction similar to the one described in the proof of Theorem 10. Let I, X, X', Y, Y', P_i, Q_i, n, c_3, δ_3 and d be as above. We transform I into an instance I' of SEX-EQUAL SMT as follows. Let $U = X \cup X' \cup S$ and $W = Y \cup Y' \cup T$ be the sets of men and women in I', respectively, where $S = \{s_1, s_2, \ldots, s_d\}$ and $T = \{t_1, t_2, \ldots, t_d\}$. The preference lists in I' are constructed as follows:

$$
\begin{array}{rll}
m_i: & P_i \quad (W \backslash P_i) & (1 \leqslant i \leqslant n), \\
m'_i: & (W) & (1 \leqslant i \leqslant (1-c_3)n), \\
s_i: & t_i \quad [W \backslash \{t_i\}] & (1 \leqslant i \leqslant d), \\
w_i: & [S] \quad Q_i \quad (X') \quad [X \backslash Q_i] & (1 \leqslant i \leqslant n), \\
w'_i: & (U) & (1 \leqslant i \leqslant (1-c_3)n), \\
t_i: & s_i \quad [U \backslash \{s_i\}] & (1 \leqslant i \leqslant d).
\end{array}
$$

Clearly in any stable matching M' in I', $(s_i, t_i) \in M'$.

Suppose firstly that I has a stable matching M such that $|M| \geqslant c_3 n$. Then we may form M' as in the proof of Theorem 10. Add (s_i, t_i) to M' $(1 \leqslant i \leqslant d)$. It may be verified that M' is stable in I'. Also the total cost of M' for the men is at most $(d+1)n + (1-c_3)n + d$. Similarly the total cost of M' for the women is at most $(2d+1)n + (1-c_3)n + d$. Hence $d(M') = |\sum_{u \in U} c_{M'}(u) - \sum_{w \in W} c_{M'}(w)| = |\sum_{u \in X} c_{M'}(u) - \sum_{w \in Y} c_{M'}(w)| \leqslant \sum_{u \in X} c_{M'}(u) + \sum_{w \in Y} c_{M'}(w) = (3d+2)n$.

On the other hand, suppose that $s^+(I) < \delta_3 c_3 n$. Now let M' be any stable matching in I'. As in the previous paragraph, the total cost of M' for the men is at most $(d+1)n + (1-c_3)n + d$. No woman $w \in Y$ is matched in M' to a man in S, so $c_{M'}(w) \geqslant d+1$. As in the proof of Theorem 10, there are $> c_3 n(1-\delta_3)$ women w in Y such that $c_{M'}(w) \geqslant (d+1) + (1-c_3)n$. Hence the total cost of M' for the women is more than

$$(d+1)n + c_3 n(1-\delta_3)(1-c_3)n + (1-c_3)n + d.$$

Thus $d(M') > \varepsilon n^2$, where ε is as defined in the proof of Theorem 10.

Therefore by Theorem 6, it is NP-hard to approximate SEX-EQUAL SMT within $\varepsilon n/(3d+2)$. □

6. Concluding remarks

It is interesting to note that the hardness results proved in this paper for approximating both MAX SMTI and MIN SMTI hold for identical restrictions on the positions of ties—there are relatively few examples in the literature of optimization problems having both maximization and minimization versions that are hard to approximate, and fewer still where this property holds for the same restrictions on the instance.

It remains open as to whether there exists an approximation algorithm for either MAX SMTI or MIN SMTI having performance ratio less than 2. However, the progress made in this paper indicates that improvements can be obtained when ties are restricted in number. One might hope for further progress when there are additional constraints in place—on the positions and lengths of ties, for example.

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